Analysis of Wave Propagation in Optical Fibers Having Core with α-Power Refractive-Index Distribution and Uniform Cladding

KATSUNARI OKAMOTO AND TAKANORI OKOSHI, MEMBER, IEEE

Abstract—**This paper describes first that a simple closed-form characteristic equation can be derived from the variational formulation of the wave propagation in an optical fiber, provided that 1) the permittivity in the core is proportional to r^α, where r is the radial coordinate and $1 < \alpha < \infty$, and 2) the cladding is uniform. The obtained equation is then solved for various permittivity (or refractive-index) profiles. The results obtained are useful both for the understanding of the dispersion characteristics and for the design of inhomogeneous optical fibers. The optimum profile for a multimode fiber is derived and discussed.**

I. Introduction

VARIOUS METHODS have been presented for the analysis of the wave propagation in radially inhomogeneous optical fibers [1]–[5]. All of them, however, required numerical analysis by computer. The propagation characteristics of an optical fiber having an α-power refractive-index profile have also been derived by using the WKB method [6]–[8]. By the WKB method, we may analyze the propagation characteristics of well-confined modes in a multimode fiber. However, the WKB method is neither applicable to a single-mode fiber nor to those modes which are close to cutoff.

The purpose of this paper is to show that a simple closed-form characteristic equation can be derived from the variational formulation given by the authors [5], provided that 1) the permittivity in the core is proportional to r^α, where r is the radial coordinate and α is a parameter between 1 and ∞, and 2) the cladding is uniform, with an arbitrary refractive index "step" (or "valley") at the core–cladding boundary as shown in Fig. 1. The equation thus obtained is applicable to a single-mode fiber and to those modes in a multimode fiber which are close to cutoff.

The obtained equation is solved for various permittivity profiles. The results obtained are helpful both for the understanding of the dispersion characteristics and for the design of inhomogeneous optical fibers. In the final part of this paper the optimum profile for a multimode fiber is derived and discussed. It is shown that for a multimode fiber, a combination of a refractive-index profile with $\alpha = 2 + y$ (where y is a parameter used by Olshansky *et al.* to represent the difference in material dispersions in the core and the cladding [9]; typically, $y = 0.3$) and a valley at the core–cladding boundary having a depth approximately one-half of the refractive-index difference in the core, is the best choice for minimizing the mode-delay difference.

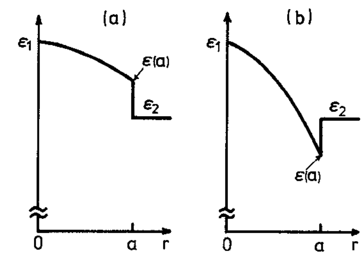

Fig. 1. Refractive-index profiles of inhomogeneous optical fibers. (a) $0 \leq \rho \leq 1$. (b) $\rho > 1$.

II. Derivation of the Characteristic Equation from Variational Formulation

A. The α-Power Refractive-Index Profile

In this paper we consider permittivity profiles expressed as

$$\varepsilon(r) = \begin{cases} \varepsilon_1[1 - 2\rho\Delta(r/a)^\alpha], & (0 \leq r \leq a) \\ \varepsilon_1[1 - 2\Delta], & (r > a) \end{cases} \tag{1}$$

where a denotes the core radius, ε_1 the permittivity at the center, Δ the relative refractive-index difference between the core axis and cladding, and ρ is a parameter representing the refractive-index step at the core–cladding interface (see Fig. 1). A smooth continuation at the core–cladding interface is expressed by $\rho = 1$. The parameter Δ may also be expressed as

$$\Delta = \frac{(n_1^2 - n_2^2)}{2n_1^2} \doteqdot \frac{n_1 - n_2}{n_1} \tag{2}$$

where n_1 and n_2 denote the refractive indices upon the axis and in the cladding, respectively.

B. Variational Formulation

The variational formulation of the wave propagation in an inhomogeneous optical fiber has been presented by the authors [5]. The same symbols will be used throughout this paper unless otherwise stated. By putting (1) into [5, eq. (17)], the functional to be dealt with in the present case is given as

Manuscript received July 3, 1975; revised December 4, 1975.
The authors are with the Department of Electronic Engineering, University of Tokyo, Tokyo 113, Japan.

$$I[R(r)] = \Phi_\beta R^2(a) + (\omega^2\varepsilon_1\mu_0 - \beta^2)\int_0^a R^2(r)r\,dr - \int_0^a \left[\left(\frac{dR}{dr}\right)^2 + \frac{m^2}{r^2}R^2\right] r\,dr - \omega^2\varepsilon_1\mu_0 2\rho\Delta \int_0^a \left(\frac{r}{a}\right)^\alpha R^2(r)r\,dr. \quad (3)$$

In the previous equation, $R(r)$ is the transverse-field function (a scalar function representing the radial variation of the wave amplitude) [5], $\Phi_\beta = a\phi_\beta$, where ϕ_β is a quantity defined as

$$\phi_\beta = \left[\frac{1}{R_{\text{clad}}(r)} \cdot \frac{dR_{\text{clad}}(r)}{dr}\right]_{r=a} \quad (4)$$

μ_0 denotes the permeability of the medium, β the propagation constant, and m the azimuthal mode number.

After some algebraic manipulations shown in Appendix I, the fourth term in (3) may be rewritten as

$$\omega^2\varepsilon_1\mu_0 2\rho\Delta \int_0^a \left(\frac{r}{a}\right)^\alpha R^2(r)r\,dr = \frac{2(\omega^2\varepsilon_1\mu_0 - \beta^2)}{(\alpha+2)}\int_0^a R^2(r)r\,dr - \frac{[(\Phi_\beta^2 - m^2) + (\omega^2\varepsilon(a)\mu_0 - \beta^2)a^2]}{(\alpha+2)} R^2(a). \quad (5)$$

Putting (5) into (3), we obtain

$$I[R(r)] = \Omega_\beta R^2(a) + \frac{\alpha}{(\alpha+2)}(\omega^2\varepsilon_1\mu_0 - \beta^2)\int_0^a R^2(r)r\,dr - \int_0^a \left[\left(\frac{dR}{dr}\right)^2 + \frac{m^2}{r^2}R^2\right] r\,dr \quad (6)$$

where

$$\Omega_\beta = \Phi_\beta + \frac{[(\Phi_\beta^2 - m^2) + (\omega^2\varepsilon(a)\mu_0 - \beta^2)a^2]}{(\alpha+2)} R^2(a). \quad (7)$$

To solve the variational problem by using Rayleigh–Ritz method, we express $R(r)$ in terms of a set of orthogonal functions $F_{m,k}(r)$ as

$$R(r) = \sum_{k=1}^{\infty} a_k F_{m,k}(r). \quad (8)$$

In an axially symmetric case, the $F_{m,k}(r)$ are expressed as

$$F_{m,k}(r) = \frac{\sqrt{2}}{a} \cdot \frac{J_m(\lambda_k r/a)}{J_m(\lambda_k)} \quad (9)$$

$$\lambda_k = \begin{cases} j_{1,k-1}, & m = 0 \\ j_{m-1,k}, & m \neq 0 \end{cases} \quad (10)$$

where $j_{n,k}$ denotes the kth root of $J_n(z) = 0$.

Putting (8) into (6) and after some computations, we may express the functional in terms of a_k as

$$I[a_1,a_2,\cdots] = \frac{2}{a^2}\Omega_\beta \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_k a_l + \frac{\alpha}{(\alpha+2)}(\omega^2\varepsilon_1\mu_0 - \beta^2)\cdot \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_k a_l \delta_{kl} - \frac{1}{a^2}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_k a_l(\lambda_k^2\delta_{kl} - 2m) \quad (11)$$

where δ_{kl} denotes Kronecker's delta. To minimize I with respect to the a_k, the following conditions must be satisfied for all k:

$$\frac{\partial I}{\partial a_k} = \frac{2}{a^2}\sum_{l=1}^{\infty} a_l S_{kl} = 0 \quad (12)$$

where

$$S_{kl} = (u^2 - \lambda_k^2)\delta_{kl} + 2(\Omega_\beta + m) \quad (13a)$$

$$u^2 = \frac{\alpha}{(\alpha+2)}(\omega^2\varepsilon_1\mu_0 - \beta^2)a^2. \quad (13b)$$

Note that variable u defined in (13b) is slightly different from the conventional one defined as $u = (\omega^2\varepsilon_1\mu_0 - \beta^2)^{1/2}a$. (The use of such a modified variable greatly simplifies equations in the present case.)

In order that a nontrivial solution of (12) exists,

$$\det(S_{kl}) = 0 \quad (14)$$

must hold. After some computations shown in the Appendix II, (14) may be rewritten as

$$(\Phi_\beta + m)\left[1 + \frac{(\Phi_\beta - m)}{(\alpha+2)}\right] + \frac{(\omega^2\varepsilon(a)\mu_0 - \beta^2)a^2}{(\alpha+2)} = \frac{uJ_{m-1}(u)}{J_m(u)}. \quad (15)$$

C. Characteristic Equation

In the previous discussions, the parameter Φ_β giving the continuity condition at the core–cladding boundary was left undetermined. In the following we calculate the parameter Φ_β to make the characteristic equation complete.

The transverse-field function in the cladding is given as

$$R_{\text{clad}}(r) = CK_m(wr/a) \quad (16)$$

where

$$w^2 = (\beta^2 - \omega^2\varepsilon_2\mu_0)a^2 \quad (17)$$

and K_m denotes the modified Hankel function of the order m. Note that w thus defined is a conventionally used symbol. From (16) and [5, eq. (13)], the parameter Φ_β is obtained as

$$\Phi_\beta = \frac{wK_m'(w)}{K_m(w)}. \quad (18)$$

Putting (18) into (15) and using the recurrence formulas of the modified Hankel functions [11], we obtain[1]

$$-\frac{wK_{m-1}(w)}{K_m(w)} - \frac{(\rho - 1/\xi_m(w))}{(\alpha + 2)} w^2 = \frac{uJ_{m-1}(u)}{J_m(u)} + \frac{(\rho - 1)}{\alpha} u^2 \tag{19}$$

where ξ_m is defined as

$$\xi_m(w) = \frac{K_m^{\,2}(w)}{K_{m-1}(w)K_{m+1}(w)}. \tag{20}$$

Equation (19) is the characteristic equation which gives the propagation constant of an inhomogeneous optical fiber having a permittivity profile given by (1). Every solution of (19) is associated with a set of linearly polarized LP_{ml} modes [10]. An LP_{ml} mode expresses two degenerate modes; one is identified as $HE_{m+1,l}$ whose θ-dependence is given as $\cos(m + 1)\theta$, and the other is TE_{0l} or $EH_{m-1,l}$ whose θ-dependence is given as $\cos(m - 1)\theta$.

In the case of a homogeneous-core fiber, we have $\alpha = \infty$. Consequently, (19) may be simplified as

$$-\frac{wK_{m-1}(w)}{K_m(w)} = \frac{uJ_{m-1}(u)}{J_m(u)} \tag{21}$$

which agrees with the characteristic equation reported previously [10], [11].

III. Cutoff Conditions

Let us examine the cutoff conditions for various propagation modes. Since the left-hand side of (19) becomes zero at cutoff (i.e., $w = 0$), u_c (the cutoff value of u) is given as the solution of

$$-\frac{J_{m-1}(u_c)}{u_cJ_m(u_c)} = \frac{(\rho - 1)}{\alpha}. \tag{22}$$

Fig. 2 shows $-(J_{m-1}/u_cJ_m)$ for several LP_{ml} modes. When α and ρ are given, u_c is obtained from the cross points of these curves and the straight line $(\rho - 1)/\alpha$. When $(\rho - 1)/\alpha \geq \frac{1}{2}$, that is

$$\rho \geq \frac{(\alpha + 2)}{2} \tag{23}$$

a cutoff exists for the fundamental (HE_{11}) mode.

We introduce here the conventional normalized frequency parameter v defined as

$$v^2 = \frac{(\alpha + 2)}{\alpha} u^2 + w^2 = \omega^2\varepsilon_1\mu_0 a^2 2\Delta. \tag{24}$$

The normalized cutoff frequency v_c is given from (24) as

$$v_c = \sqrt{1 + 2/\alpha} \cdot u_c. \tag{25}$$

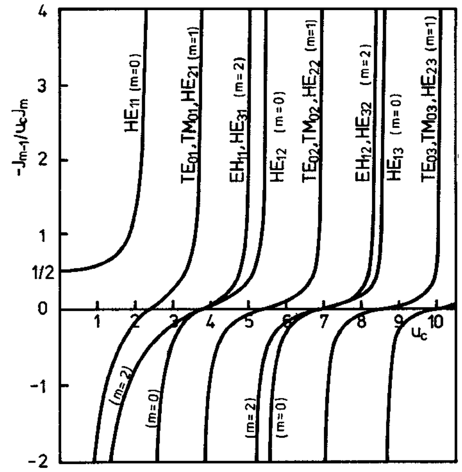

Fig. 2. Plots of $-(J_{m-1}/u_cJ_m)$ versus u_c for various modes.

When $\rho = 1$ or $\alpha = \infty$, the right-hand side of (22) vanishes, and hence $u_c = j_{m-1,k}$. Thus (25) leads to

$$v_c = \begin{cases} \sqrt{1 + 2/\alpha}\, j_{m-1,k}, & \rho = 1 \text{ and } \alpha \neq \infty \\ j_{m-1,k}, & \alpha = \infty. \end{cases} \tag{26}$$

The single-mode limit of the fiber, that is, the cutoff of the second lowest (TE_{01}) mode, is given as

$$v < \sqrt{1 + 2/\alpha} \cdot u_{cs} \tag{27}$$

where u_{cs} is the smallest value of the solutions of (22) for $m = 1$. Fig. 2 shows that, as the parameter $(\rho - 1)/\alpha$ $(= -J_{m-1}/u_cJ_m)$ increases, u_{cs} also increases, and the range of the normalized frequency for the single-mode operation is widened. From (23) and the previous discussions, the largest value of v for the single-mode operation, within the limit that no cutoff is present for the HE_{11} mode, may be obtained when

$$\rho = \frac{(\alpha + 2)}{2} - \varepsilon \tag{28}$$

where ε is an arbitrarily small quantity.

IV. Optimum Profile for a Multimode Fiber

A. Propagation Constant and Dispersion Characteristics

The propagation constant β, group delay $\tau(\omega)$, and dispersion $\sigma(\omega)$ may be expressed, by using the solution (u–v relation) of the characteristic equation (19), as

$$\beta = kn_1(1 - 2x\Delta)^{1/2}, \quad (k = \omega/c,\ c\text{: light velocity}) \tag{29}$$

$$\tau \triangleq \frac{d\beta}{d\omega} = \frac{N_1}{c}\frac{[1 - \Delta(1 + y/4)Qx]}{(1 - 2x\Delta)^{1/2}} \tag{30}$$[2]

and

$$\sigma \triangleq \omega\frac{d^2\beta}{d\omega^2} = \frac{\tau(x)}{N_1}\lambda^2\frac{d^2n_1}{d\lambda^2} + \Delta\frac{N_1}{c}v\frac{d^2}{dv^2}[v(1 - x)] \tag{31}$$

[1] An approximation $(1 - \varepsilon(a)/\varepsilon_2) \simeq 0$ is used in deriving (19). Therefore, when $\rho \neq 1$, that is, when a step or valley is present at the core–cladding boundary, a small amount of error will be introduced by using (19). However, the first-order effect of the refractive-index step or valley at the core–cladding boundary is represented by the terms $(\rho - 1/\xi_m)$ and $(\rho - 1)$ in (19).

[2] Using the variational expression of the propagation constant, we can directly obtain the same equation for τ (see Appendix III).

where

$$N_1 = \frac{d(kn_1)}{dk} \tag{32}$$

$$y = -\frac{2n_1}{N_1}\frac{\lambda}{\Delta}\frac{d\Delta}{d\lambda} \tag{33}$$

$$Q = \frac{2v}{u}\frac{du}{dv} = \frac{4}{(\alpha + 2)}\left\{1 + \frac{\alpha(1 - \xi_m) + 2(\rho - 1)\xi_m}{2x(1 - \xi_m/\zeta_m)}\right\} \tag{34}$$

$$\zeta_m = \frac{J_m^{\,2}(u)}{J_{m-1}(u)J_{m+1}(u)} \tag{35}$$

and

$$x = \frac{(\alpha + 2)}{\alpha}\frac{u^2}{v^2}. \tag{36}$$

Among the previously listed parameters, N_1 is the group index, y is a parameter representing the difference in material dispersion in the core and the cladding [9] (typically, $y = 0.3$), and x is a parameter representing the order of modes. From (24) we have $0 \leq x \leq 1$ and

$$x = \begin{cases} 0, & \text{for the lowest mode} \\ 1, & \text{for the highest mode.} \end{cases} \tag{37}$$

The first and second terms of (31) represent the material dispersion and waveguide dispersion, respectively.

B. Mode-Delay Differences

Since Δ is much smaller than unity, we may expand (30) into a power series in terms of Δ to obtain the expression for the delay difference between the lowest mode and the xth mode, as

$$\delta\tau \triangleq \tau(x) - \tau(0) = \frac{N_1}{c}\left\{\Delta\left[1 - \left(1 + \frac{y}{4}\right)Q\right]x + \Delta^2\left[\frac{3}{2} - \left(1 + \frac{y}{4}\right)Q\right]x^2 + 0(\Delta^3)\right\}. \tag{38}$$

C. Optimum Parameters for a Multimode Fiber

We find from (38) that when

$$Q = Q_{\text{opt}} = \frac{1}{1 + y/4} \tag{39}$$

$\delta\tau$ becomes very small. However, since α is given as a function of not only Q but also x, ξ_m, and ζ_m [(34)], the optimum value of α varies from mode to mode. Hence, generally, we cannot obtain the optimum α for all modes. However, if we control the value of ρ, we may have a common optimum value of α regardless of the mode, as discussed in the following.

For those modes which are far from cutoff and well confined in the core, $w \gg 1$ and the left-hand side of (19) becomes negatively very large, and u in the right-hand side approaches $j_{m,k}$ (the kth root of $J_m(z) = 0$). For those modes we may write

$$\xi_m \fallingdotseq 1 \qquad \zeta_m \fallingdotseq 0 \tag{40}$$

and (34) is simplified as

$$\alpha_{\text{opt}} = \frac{4}{Q_{\text{opt}}} - 2 = 2 + y \tag{41}$$

regardless of the mode. For those modes which are close to cutoff, u is given approximately as the solution of (22). Fig. 2 shows that when $(\rho - 1)/\alpha > \frac{1}{2}$, the cutoff value u_c is nearly equal to $j_{m,k}$. Therefore, in this particular case $\zeta_m \fallingdotseq 0$ for all modes, and (41) also holds. Thus, when

$$\alpha = \alpha_{\text{opt}} = 2 + y \tag{42}$$

$$\rho > \alpha/2 + 1 = 2 + y/2 \tag{43}$$

the mode-delay difference $\delta\tau$ becomes very small for all modes.

In the previous discussions, the condition for ρ has been obtained as an inequality. We should note, however, that in practice if ρ is excessively large and hence a very deep valley is present at the core–cladding boundary, many leaky modes will propagate with relatively small loss and the actual delay difference will increase. A practical conclusion, therefore, will be that the value of ρ slightly larger than $(2 + y/2)$ is the best choice.

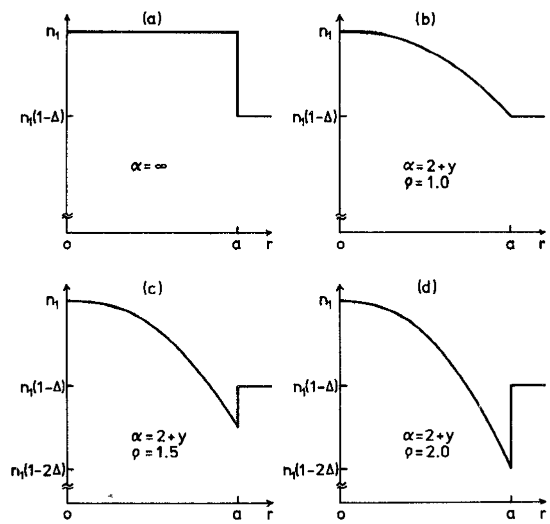

Fig. 3. Various refractive-index profiles used in the numerical analysis.

V. Discussions

1) The group delay $\tau(x)$ has been computed by using (30)–(36) for four refractive-index profiles illustrated in Fig. 3 and are shown in Fig. 4(a) ($\alpha = \infty$, $\rho = 1$), Fig. 4(b) ($\alpha = 2 + y$, $\rho = 1$), Fig. 4(c) ($\alpha = 2 + y$, $\rho = 1.5$), and Fig. 4(d) ($\alpha = 2 + y$, $\rho = 2.0$). It will be found that when $\alpha = 2 + y$ and $\rho \fallingdotseq 2 + y/2$ [Fig. 4(d)], mode-delay differences are reduced dramatically.

2) Analysis of fibers having such a "valley" have been reported by Kawakami *et al.* (for a three-layer slab model [12] and a three-layer axially symmetric model [13]) and Furuya *et al.* (for a three-layer slab model [14] and a slab

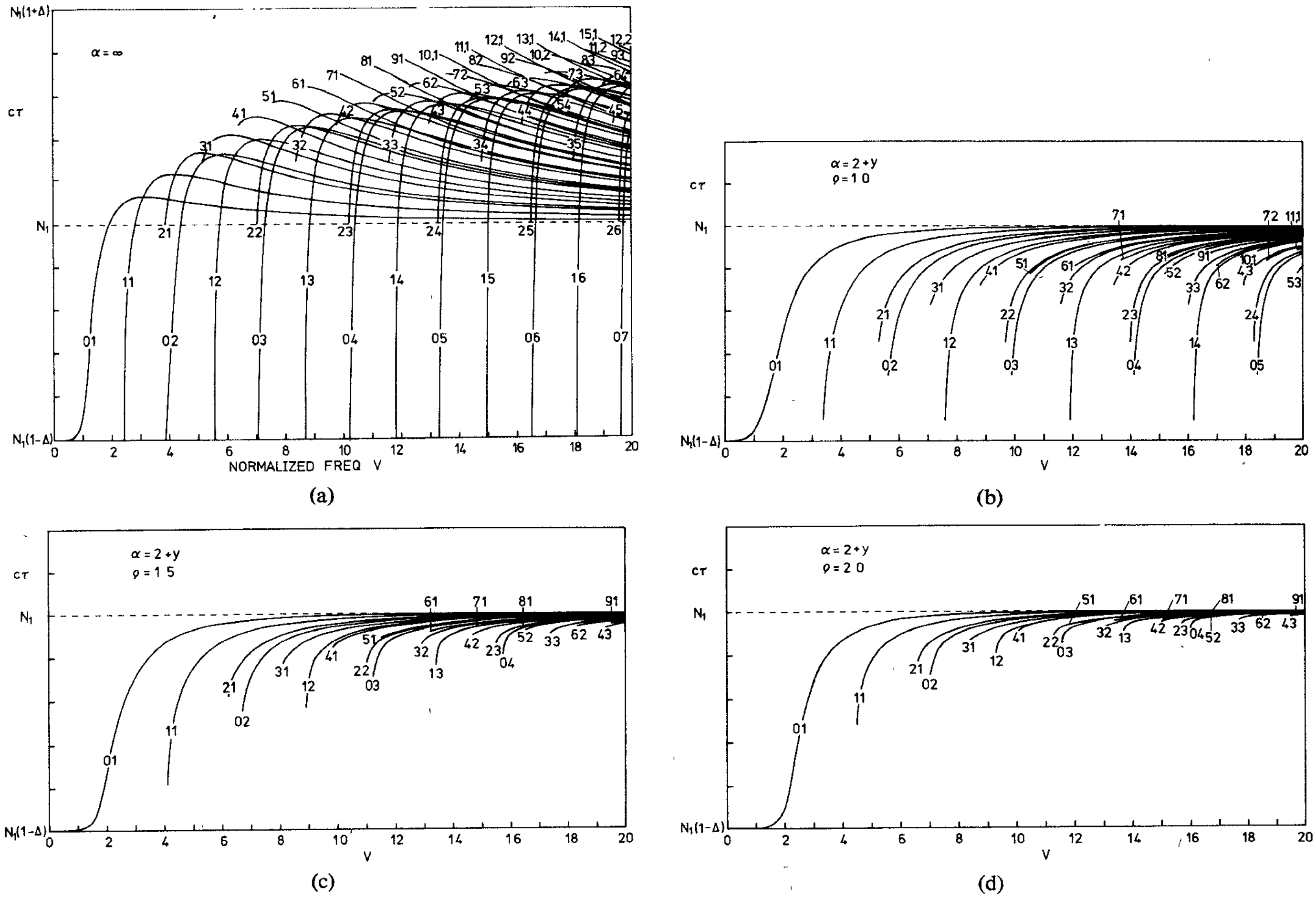

Fig. 4. Group delay $\tau(x)$ of inhomogeneous optical fibers having index profiles shown in Fig. 3.

model with a quadratic core profile [15]). Kawakami *et al.* aimed at broadening the bandwidth of a single-mode fiber by obtaining a waveguide dispersion which cancels the material dispersion, whereas Furuya *et al.* aimed at the rejection of higher modes in a multimode fiber. The effectiveness of such a "valley" has also been verified experimentally [16].

3) The optimum value of ρ has not yet been determined precisely. Moreover, the optimization described in this paper is limited to the α-power distribution in the core. The real optimization of the refractive-index profile without such constraints is now being investigated; this will be reported elsewhere.

VI. Conclusion

The results obtained may be summarized as follows.

1) The characteristic equation for an inhomogeneous optical fiber having a permittivity profile given by (1) has been obtained analytically as (19). As the parameter $(\rho - 1)/\alpha$ increases, the maximum normalized frequency giving the single-mode limit increases. However, when $(\rho - 1)/\alpha > \frac{1}{2}$, a cutoff (low-frequency limit) appears for the fundamental mode.

2) From (19) we obtain the propagation constant, group delay, and waveguide dispersion as (29)–(31). Numerical solutions of these propagation characteristics for various refractive-index profiles are shown.

3) The combination of $\alpha = 2 + y$ and $\rho \fallingdotseq 2 + y/2$ is the optimum choice for a multimode fiber. In such a case the mode-delay differences can be very small for all modes including those which are close to cutoff.

Appendix I

Derivatation of (5)

The transverse-field function satisfies the following differential equation and the continuity condition [5]:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left[\omega^2\varepsilon(r)\mu_0 - \beta^2 - \frac{m^2}{r^2}\right]R(r) = 0 \quad \text{(A1)}$$

$$\left[\frac{r}{R(r)}\frac{dR}{dr}\right]_{r=a} = \left[\frac{r}{R_{\text{clad}}(r)}\frac{dR_{\text{clad}}(r)}{dr}\right]_{r=a} \triangleq \Phi_\beta. \quad \text{(A2)}$$

Multiplying (A1) by $r^2(dR/dr)$ and integrating with respect to r, we obtain

$$\int_0^a r\frac{dR}{dr}\frac{d}{dr}\left(r\frac{dR}{dr}\right)dr + (\omega^2\varepsilon_1\mu_0 - \beta^2)\int_0^a r^2R\frac{dR}{dr}dr - m^2\int_0^a R\frac{dR}{dr}dr - \omega^2\varepsilon_1\mu_0 2\rho\Delta\int_0^a\left(\frac{r}{a}\right)^\alpha r^2R\frac{dR}{dr}dr = 0. \quad \text{(A3)}$$

Partial integration of each term of (A3) and use of (A2) lead to

$$\omega^2\varepsilon_1\mu_0 2\rho\Delta \int_0^a \left(\frac{r}{a}\right)^\alpha R^2(r)r\,dr$$
$$= \frac{2(\omega^2\varepsilon_1\mu_0 - \beta^2)}{(\alpha + 2)} \int_0^a R^2(r)r\,dr$$
$$- \frac{[(\Phi_\beta{}^2 - m^2) + (\omega^2\varepsilon(a)\mu_0 - \beta^2)a^2]}{(\alpha + 2)} R^2(a). \quad \text{(A4)}$$

Appendix II

Derivatation of (15)

From (13a) and (14), we obtain

$$\begin{vmatrix} \Omega + (u^2 - \lambda_1{}^2) & \Omega & \cdots & \Omega \\ \Omega & \Omega + (u^2 - \lambda_2{}^2) & \cdots & \Omega \\ \vdots & \vdots & & \vdots \\ \Omega & \Omega & \cdots & \Omega + (u^2 - \lambda_N{}^2) \end{vmatrix} = 0 \quad \text{(A5)}$$

where

$$\Omega = 2(\Omega_\beta + m). \quad \text{(A6)}$$

Equation (A5) may be rewritten to

$$\frac{1}{\Omega} = -\sum_{k=1}^{N} \frac{1}{(u^2 - \lambda_k{}^2)}. \quad \text{(A7)}$$

Putting (10) into (A7), setting $N \to \infty$, and using one of the Bessel-function formulas [17], we obtain

$$-\sum_{k=1}^{\infty} \frac{1}{(u^2 - \lambda_k{}^2)} = \begin{cases} -\dfrac{J_0(u)}{2uJ_1(u)}, & m = 0 \\ \dfrac{J_m(u)}{2uJ_{m-1}(u)}, & m \neq 0. \end{cases} \quad \text{(A8)}$$

Hence, from (A6) and (A8) we have

$$\Omega_\beta + m = \frac{uJ_{m-1}(u)}{J_m(u)}. \quad \text{(A9)}$$

Appendix III

Variational Expression of Group Delay

Variational equation (3) of the wave propagation is equivalent to

$$\beta^2 = \frac{\int_0^\infty k^2n^2(r)\Phi^2(r)r\,dr - \int_0^\infty \left\{\left(\frac{d\Phi}{dr}\right)^2 + \frac{m^2}{r^2}\Phi^2\right\} r\,dr}{\int_0^\infty \Phi^2(r)r\,dr} \quad \text{(A10)}$$

which should be stationary with respect to a small variation of Φ. Therefore, we may find $d\beta/dk$ by differentiating (A10) only where k appears explicitly. Thus, as shown by Case [18],

$$\frac{\beta}{k}\frac{d\beta}{dk} = \frac{\int_0^\infty n^2(r)\Phi^2(r)r\,dr + \int_0^\infty kn\frac{dn}{dk}\Phi^2(r)r\,dr}{\int_0^\infty \Phi^2(r)r\,dr}. \quad \text{(A11)}$$

When the relative refractive-index difference Δ and the core radius a are given, we may determine the propagation constant β from the characteristic equation (19), and further determine the values of u, w, and the expansion coefficients a_k $(k = 1,2,\cdots)$. Using the a_k we may express the transverse field in the core as

$$R(r) = R_0 \sum_{k=1}^{\infty} \frac{J_m(\lambda_k r/a)}{(u^2 - \lambda_k{}^2)J_m(\lambda_k)}. \quad \text{(A12)}$$

Putting (A12), (16), and (5) into (A11), we obtain (30).

Acknowledgment

The authors wish to thank Prof. S. Kawakami of Tohoku University for his helpful suggestions.

References

[1] P. J. B. Clarricoats and K. B. Chan, "Electromagnetic-wave propagation along radially inhomogeneous dielectric cylinders," *Electron. Lett.*, vol. 6, p. 649, Oct. 1970.
[2] H. Kirchhoff, "Wave propagation along radially inhomogeneous glass fibers," *A.E.Ü.*, vol. 27, no. 1, p. 13, 1973.
[3] J. G. Dil and H. Blok, "Propagation of electromagnetic surface waves in a radially inhomogeneous optical waveguide," *Opt. Electron.*, vol. 5, p. 415, 1973.
[4] M. Matsuhara, "Analysis of TEM modes in dielectric waveguides, by a variational method," *J.O.S.A.*, vol. 63, p. 1514, Dec. 1973.
[5] T. Okoshi and K. Okamoto, "Analysis of wave propagation in inhomogeneous optical fibers using a variational method," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-22, p. 938, Nov. 1974.
[6] D. Gloge and E. A. J. Marcatili "Multimode theory of graded-core fibers," *B.S.T.J.*, vol. 52, p. 1563, Nov. 1973.
[7] R. Olshansky, "Mode coupling effects in graded-index optical fibers," *Appl. Opt.*, vol. 14, p. 935, April 1975.
[8] D. Gloge, "Propagation effects in optical fibers," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-23, p. 106, Jan. 1975.
[9] R. Olshansky *et al.*, "Material effects on minimizing pulse broadening," a paper read at the Topical Meeting on Optical Fiber Transmission, January 1975, at Williamsburg, VA.
[10] D. Gloge, "Weakly guiding fibers," *Appl. Opt.*, vol. 10, no. 10, p. 2252, 1971.
[11] A. W. Snyder, "Asymptotic expressions for eigenfunctions and eigenvalues of a dielectric or optical waveguide," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-17, p. 30, 1969.
[12] Y. Ohtaka, S. Kawakami, and S. Nishida, "Transmission characteristic of a multi-layer dielectric slab optical waveguide with strongly evanescent wave layers," *Trans. Inst. Electron. Commun. Eng. Japan*, vol. 57-C, p. 187, 1974.
[13] S. Kawakami and S. Nishida, "Characteristics of a double clad optical fiber with a low-index inner cladding," *IEEE J. Q. E.*, vol. QE-10, no. 12, 1974.
[14] Y. Suematsu and K. Furuya, "Quasi-guided modes and related radiation losses in optical dielectric waveguides with external higher index surroundings," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-23, p. 170, Jan. 1975.
[15] Y. Suematsu *et al.*, "Refractive index distribution and group delay characteristics in multimode dielectric optical waveguides," *Trans. Inst. Electron. Commun. Eng. Japan*, vol. 52-C, p. 289, Sept. 1974.
[16] S. Onoda *et al.*, "Transmission characteristics of *W*-type optical fiber," Paper of Technical Group on Optoelectronics and Quantum Electronics, No. OQE 74-46, p. 45, 1974.
[17] G. N. Watson, *Theory of Bessel Function.* New York: Cambridge Univ. Press, 1922, p. 498.
[18] K. M. Case, "On wave propagation in inhomogeneous media," *J. Math. Phys.*, vol. 13, p. 360, March 1972.